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On Observables[†]

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We propose a simple set-theoretic model of a generalized probability space admitting intrinsic incompatible events and incompatible observables. It is a coproduct in the category $\mathcal{D}$ of D-posets and D-homomorphisms each factor of which is a classical Kolmogorovian probability space. Since classical events, random functions, and probability measures can be treated within $\mathcal{D}$ in a canonical way, the Kolmogorovian model becomes a special case. We show that σ-additivity and other σ-notions can be replaced in a natural way by sequential continuity.

1. INTRODUCTION

Example 1.1. Let $(\Omega, \mathbb{A}, p)$ be a probability space and let $f: \Omega \to R$ be a random variable. Each elementary event $\omega \in \Omega$ represents an "atomic realization of a random experiment" and $f(\omega)$ is the real number assigned to ω in the corresponding measurement. Further, for each Borel set B of real numbers, $f^{\leftarrow}(B)$ is the set of all elementary events for which the measurement terminates in B and $p(f^{\leftarrow}(B))$ is its probability.

Now, assume that we start with a larger set X carrying a σ-ring $\mathbb{S}$ of its subsets, a measurable map $g: X \to R$, and a (finite) measure m such that $\Omega \subset X$, $\mathbb{A} \subset \mathbb{S}$ $0 < m(\Omega)$, and for each $A \in \mathbb{A}$ we have $p(A) = m(A)/m(\Omega)$. Even though m need not be a probability, we can view $(\Omega, \mathbb{A}, p)$ and $f = g|\Omega$ as a partial experiment in a broader setting represented by $(X, \mathbb{S}, m)$, where Ω and $\mathbb{A}$ are certain limitations (restrictions) imposed for some reasons on the experiment and the corresponding measurement, and m is a "gauge."

Of course, we can consider families $\{(\Omega_t, \mathbb{A}_t, p_t); t \in T\}$, $\{f_t = g \mid \Omega_t; t \in T\}$ of such partial experiments within the same $(X, \mathbb{S}, m)$. Then events

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belonging to different $\mathbb{A}_t$ and different random variables f_t need not be "compatible." Indeed, it can even happen that a certain event A belongs to $\mathbb{A}_s$ and $\mathbb{A}_t$ for $s, t \in T$, $s \neq t$, and $p_s(A) \neq p_t(A)$.

Motivated by this model situation, we propose generalizations of the notions of a probability space and a random function.

Observation 1.2. Example 1.1 allows for various modifications and ramifications. For example:

(i) The measure m can be infinite. Then, if $m(\Omega)$ is infinite, we have to define $p(A) = 0$ whenever $m(A)$ is finite, and otherwise define $p(A)$ in some consistent way.
(ii) If $\Omega \neq \emptyset$ and $m(\Omega) = 0$, still p and m can be related in some consistent way.
(iii) The family Ω_t, $t \in T$, can have some semilattice property.
(iv) The family $\{\mathbb{A}_t;\ t \in T\}$ can be specified more explicitly, e.g., $\mathbb{A}_t = \{A \cap \Omega_t | A \in \mathbb{S}\}$.

Example 1.1 can be analyzed within the theory of quantum logics (cf. Pták and Pulmannová, 1991). In fact, we shall proceed this way, but we will restrict ourselves to rather simple quantum logics and we will use the apparatus of D-posets.

First, each $(\Omega_t, \mathbb{A}_t, p_t)$ is a classical original probability space and f_t is a classical random variable. In this way we split up our initial experiment into a family of incompatible experiments and we study each of them within the classical Kolmogorovian probability theory. Second, according to Kôpka and Chovanec (1994), events, observables, and probability measures can be described within the category $\mathcal{D}$ so that $\mathbb{A}_t$ becomes a D-poset and $f_t^{\leftarrow}$ and p_t become D-homomorphisms. Third, we paste the family $\{(\Omega_t, \mathbb{A}_t, p_t);\ t \in T\}$ as a coproduct. This yields a simple generalization of the classical Kolmogorovian model.

Recall that a D-poset is a quintuple $(X, \leqq_X, \ominus_X, 0_X, 1_X)$, where X is a set, $\leqq_X$ is a partial order on X, 0_X is the least element, 1_X is the greatest element, $\ominus_X$ is a partial operation on X such that $a \ominus_X b$ is defined iff $b \leqq_X a$, and the following axioms are assumed:

(D1) $a \ominus_X 0_X = a$ for each $a \in X$
(D2) If $c \leqq_X b \leqq_X a$, then $a \ominus_X b \leqq_X a \ominus_X c$ and $(a \ominus_X c) \ominus_X (a \ominus_X b) = b \ominus_X c$.

If no confusion arises, the quintuple $(X, \leqq_X, \ominus_X, 0_X, 1_X)$ is condensed to X; sometimes the index denoting the underlying set is omitted.

A map h on a D-poset X into a D-poset Y which preserves the D-poset structure is said to be a *D-homomorphism*. In addition, if h preserves all existing suprema of nondecreasing sequences in X, then h is said to be a σ-*D-homomorphism*. As observed by F. Chovanec and F. Kôpka, for Boolean algebras D-homomorphisms and Boolean homomorphisms coincide. A σ-D-homomorphism of the σ-field of all Borel sets of the real line R into a D-poset Y is called an *observable on Y*, and a σ-D-homomorphism of a D-poset Y into the interval $[0, 1]$ (carrying the usual difference and order) is called a *state on Y.*

2. Δ-SUM AND $\mathcal{D}$-COPRODUCT

Let $\{X_t;\ t \in T\}$ be a family of sets. We do not exclude the case when each X_t is a fixed set, e.g., the real line, or a subset of a given set X. Sometimes we need to treat the sets X_t as mutually disjoint. Then the elements of X_t will be denoted by (x, t) and hence $s \neq t$ implies $(x, s) \neq (x, t)$; X_t will be replaced by $X(t)$. In some cases we simply assume that $X_x \cap X_t = \emptyset$ whenever $s \neq t$.

Let Ω be a set and let $\mathbb{A}$ be a field of its subsets carrying the usual partial order (inclusion) and set operations. Clearly, $(\Omega, \mathbb{A})$ can be considered as a D-poset $(\mathbb{A}, \subseteq, \ominus_{\mathbb{A}}, \emptyset, \Omega)$ where the partial operation $\ominus_{\mathbb{A}}$ is defined as follows: for $A, B \in \mathbb{A}$, $A \ominus_{\mathbb{A}} B$ is defined iff $A \supseteq B$ and then we put $A \ominus_{\mathbb{A}} B = A \backslash B$.

Let f be a measurable map of a field $(\Omega_1, \mathbb{A}_1)$ into a field $(\Omega_2, \mathbb{A}_2)$. Clearly, $f^{\leftarrow}\colon \mathbb{A}_2 \to \mathbb{A}_1$ is a D-homomorphism of $(\mathbb{A}_2, \subseteq, \ominus_{\mathbb{A}_2}, \emptyset, \Omega_2)$ into $(\mathbb{A}_1, \subseteq, \ominus_{\mathbb{A}_1}, \emptyset, \Omega_1)$.

Construction 2.1. Let $\{(\Omega_t, \mathbb{A}_t);\ t \in T\}$ be a family of fields of sets. For each $t \in T$, let $\Omega(t) = \Omega_t \times \{t\}$, let $\mathbb{A}(t) = \{\{(\omega, t) \in \Omega(t) | \omega \in A\} | A \in \mathbb{A}_t\}$; then $(\Omega_t, \mathbb{A}_t)$ and $(\Omega(t), \mathbb{A}(t))$ are isomorphic and $\Omega(s) \cap \Omega(t) = \emptyset$ whenever $s, t \in T$, $s \neq t$. Let $\Omega = \cup_{t \in T} \Omega(t)$, let $\mathbb{S}(t) = \{A \in \mathbb{A}(t) | A \neq \emptyset, A \neq \Omega(t)\}$, and let $\mathbb{S} = \{\emptyset, \Omega\} \cup (\cup_{t \in T} \mathbb{S}(t))$. Define a partial operation $\ominus_{\mathbb{S}}$ on $\mathbb{S}$, partially ordered by inclusion, as follows: For $A, B \in \mathbb{S}$, $A \ominus_{\mathbb{S}} B$ is defined iff $A \supseteq B$ and then:

(i) Put $\Omega \ominus_{\mathbb{S}} \Omega = \emptyset$, $\Omega \ominus_{\mathbb{S}} \emptyset = \Omega$, and $\emptyset \ominus_{\mathbb{S}} \emptyset = \emptyset$.
(ii) For each $A \in \mathbb{S}(t)$, $t \in T$, put $\Omega \ominus_{\mathbb{S}} A = \Omega_t \backslash A$ and $A \ominus_{\mathbb{S}} \emptyset = A$.
(iii) For $A, B \in \mathbb{S}(t)$, $A \supseteq B$, $t \in T$, put $A \ominus_{\mathbb{S}} B = A \backslash B$.

For each $t \in T$, define a map $\kappa_t\colon \mathbb{A}_t \to \mathbb{S}$ as follows: $\kappa_t(\Omega_t) = \Omega$ and, for $A \neq \Omega_t$, put $\kappa_t(A) = \{(\omega, t) \in \Omega(t) | \omega \in A\}$.

Proposition 2.2. Let $\{(\Omega_t, \mathbb{A}_t);\ t \in T\}$ be a family of fields of sets. Then $(\mathbb{S}, \subseteq, \ominus_{\mathbb{S}}, \emptyset, \Omega)$ is a D-poset. For each $t \in T$, κ_t is an D-isomorphism onto a D-poset subspace of $(\mathbb{S}, \subseteq, \ominus_{\mathbb{S}}, \emptyset, \Omega)$.

Proof. The straightforward calculations are omitted.

Definition 2.3. We shall call $(\mathbb{S}, \subseteq, \ominus_{\mathbb{S}}, \emptyset, \Omega)$ the Δ-*sum* of $\{(\Omega_t, \mathbb{A}_t); t \in T\}$; it will be denoted by $\Delta_{t\in T}(\Omega_t, \mathbb{A}_t)$. Maps $\kappa_t\colon \mathbb{A}_t \to \mathbb{S}$, $t \in T$, are called *coprojections*.

Observe that $\mathbb{S}$ is in fact the *horizontal sum* of its almost disjoint subspaces $\kappa_t(\mathbb{A}_t)$, $t \in T$ (Dvurečenskij, 1993). We claim that $\Delta_{t\in T}(\Omega_t, \mathbb{A}_t)$ is the coproduct in the category of D-posets and D-homomorphisms and hence it has some useful properties.

Let $\{(X_t, \leqq_t, \ominus_t, 0_t, 1_t); t \in T\}$ be a family of D-posets. Recall that a D-poset $(X, \leqq_X, \ominus_X, 0_X, 1_X)$ together with D-homomorphisms $\{\kappa_t\colon X_t \to X; t \in T\}$, called *coprojections*, is the *coproduct* of $\{(X_t, \leqq_t, \ominus_t, 0_t, 1_t); t \in T\}$ if, whenever $(U, \leqq_U, \ominus_U, 0_U, 1_U)$ is a D-poset and $\{\varphi_t\colon X_t \to U; t \in T\}$ are D-homomorphisms, then there is a unique D-morphism $\Phi\colon X \to U$ such that $\Phi \circ \kappa_t = \varphi_t$ for each $t \in T$.

Construction 2.4. For each $t \in T$, let $(X_t, \leqq_t, \ominus_t, 0_t, 1_t)$ be a D-poset, let $Y_t = \{(x, t) | x \neq 0_t, x \neq 1_t\}$, and let $Y = \{0, 1\} \cup (\cup_{t\in T} Y_t)$. Define a relation $\leqq$ on Y as follows:

(i) $0 \leqq y \leqq 1$ for each $y \in Y$.
(ii) $v \leqq u$ whenever $u = (x, t)$, $v = (y, t)$ for some $t \in T$ and $y \leqq_t x$.

Clearly, $\leqq$ is a partial order, 0 is the least element in Y, and 1 is the greatest element in Y. Define a partial operation $\ominus$ on Y as follows: $x \ominus y$ is defined iff $y \leqq x$ and then:

(iii) Put $1 \ominus 1 = 0$, $1 \ominus 0 = 1$, $0 \ominus 0 = 0$.
(iv) For each $(x, t) \in Y_t$, $t \in T$, put $1 \ominus (x, t) = (1_t, \ominus_t, x, t)$ and $(x, t) \ominus 0 = (x, t)$.
(v) For each $(x, t), (y, t) \in Y_t$, $y \leqq_t x$, $t \in T$, put $(x, t) \in (y, t) = 0$ whenever $x = y$ and $(x, t) \ominus (y, t) = (x, \ominus_t y, t)$ otherwise.

For each $t \in T$, define a map $\kappa_t\colon X_t \to Y$ as follows: $\kappa_t(0_t) = 0$, $\kappa_t(1_t) = 1$, and, for $x \neq 0_t$, $x \neq 1_t$, put $\kappa_t(x) = (x, t)$.

The proofs of the next two assertions are straightforward and are omitted.

Proposition 2.5. Let $\{(X_t, \leqq_t, \ominus_t, 0_t, 1_t); t \in T\}$ be a family of D-posets. Then $(Y, \leqq, \ominus, 0, 1)$ is a D-poset and, for each $t \in T$, κ_t is a D-isomorphism of $(X_t, \leqq_t, \ominus_t, 0_t, 1_t)$ onto a D-poset subspace of $(Y, \leqq, \ominus, 0, 1)$.

Corollary 2.6. Let $\{(X_t, \leqq_t, \ominus_t, 0_t, 1_t); t \in T\}$ be a family of D-posets. Then $(Y, \leqq, \ominus, 0, 1)$ together with $\{\kappa_t; t \in T\}$ is its coproduct. In particular, if $\{(\Omega_t, \mathbb{A}_t); t \in T\}$ is a family of fields of sets, then $\Delta_{t\in T}(\Omega_t, \mathbb{A}_t)$ together with $\{\kappa_t; t \in T\}$ is its coproduct.

Definition 2.7. Let $\{(X_t, \leqq_t, \ominus_t, 1_t);\ t \in T\}$ be a family of D-posets and let $(Y, \leqq, \ominus, 0, 1)$ together with $\{\kappa_t;\ t \in T\}$ be its coproduct. Let Ψ be a D-morphism of $(Y, \leqq, \ominus, 0, 1)$ into a D-poset $(U, \leqq_U, \ominus_U, 0_U, 1_U)$. Then $\{\Psi \circ \kappa_t;\ t \in T\}$ is said to be the *spectrum* of Ψ.

Construction 2.8. Let $\{(\Omega_t, \mathbb{A}_t, p_t);\ t \in T\}$ be a family of probability spaces. Let P be the unique D-homomorphism of $(\mathbb{S}, \subseteq, \ominus_{\mathbb{S}}, \emptyset, \Omega) = \Delta_{t \in T}(\Omega_t, \mathbb{A}_t)$ into $[0, 1]$ such that $P \circ \kappa_t = p_t$, $t \in T$. Denote by $(\mathbb{D}, \subseteq, \ominus_{\mathbb{D}}, \emptyset, D)$ the Δ-sum $\Delta_{t \in T}(R_t, \mathbb{B}_t)$ in which each factor is the real line R carrying the σ-field $\mathbb{B}$ of Borel sets. Let $\{\phi_t\colon \mathbb{B}_t \to \mathbb{D};\ t \in T\}$ be the corresponding coprojections. For each $t \in T$, let f_t be a random variable on $(\Omega_t, \mathbb{A}_t, p_t)$. This yields a map f of $\Omega = \cup_{t \in T}\Omega(t)$ into $D = \cup_{t \in T}R(t)$ defined by $f(\omega, t) = (f_t(\omega), t)$, $\omega \in \Omega_t$, $t \in T$. For each $t \in T$, the preimage $f_t^{\leftarrow}$ of f_t is a (sequentially continuous) Boolean homomorphism and hence a D-homomorphism of $\mathbb{B}_t$ into $\mathbb{A}_t$. The family $\{f_t^{\leftarrow};\ t \in T\}$ yields a generalized preimage f^{∇} of f which maps $\mathbb{D}$ into $\mathbb{S}$. Indeed, let $M \in \mathbb{D}$. Then there are $t \in T$ and $B \in \mathbb{B}_t$ such that $M = \phi_t(B)$. Define $f^{\nabla}(M) = \kappa_t(f_t^{\leftarrow}(B))$. Then f^{∇} is well-defined and maps M into the smallest element of $\mathbb{S}$ containing the subset $f_t^{\leftarrow}(B)$ of Ω. Observe that $f^{\nabla}(M) = \Omega$ iff $f_t^{\leftarrow}(B) = \Omega_t$. It is a D-homomorphism of $\Delta_{t \in T}(R_t, \mathbb{B}_t)$ into $\Delta_{t \in T}(\Omega_t, \mathbb{A}_t)$. We shall show that both $\Delta_{t \in T}(\Omega_t, \mathbb{A}_t)$ and $\Delta_{t \in T}(R_t, \mathbb{B}_t)$ can be equipped with a canonical sequential convergence such that P, f^{∇}, and hence $P \circ f^{\nabla}$ become sequentially continuous.

Our probability model consists of $\Delta_{t \in T}(\Omega_t, \mathbb{A}_t)$ as a *generalized field of events*, P as a *generalized probability*, f as a *generalized random variable*, and f^{∇} as a *generalized observable*. We shall show that σ-additivity and other σ-notions can be replaced in a natural way by sequential continuity.

Due to the fact that $\mathbb{S} = \Delta_{t \in T}(\Omega_t, \mathbb{A}_t)$ is a coproduct in $\mathcal{D}$, it is possible first to study "factors" $(\Omega_t, \mathbb{A}_t, p_t)$ and f_t within $\mathcal{D}$ and then to "paste" the corresponding results to get results concerning $\mathbb{S}$, P, and f.

Observation 2.9. Observe that if $\Omega(t)$ is partitioned into disjoint hypotheses, then for events in $\mathbb{S}(s)$, $s \neq t$, the total probability rule and the Bayes formula do not hold. Hence our model has a quantum nature. The famous two-slit experiment can be accommodated into the model, e.g., as follows. Starting with $(X, \mathbb{S}, m)$, let $(\Omega_i, \mathbb{A}_i, p_i)$, $i = 1,2$, represent the experiment when on the first screen only the first, resp. the second, slit is open. Let $(\Omega_3, \mathbb{A}_3, p_3)$ represent the experiment when both slits are open. For $\Omega_1 \cap \Omega_2 = \emptyset$, $\Omega_1 \cup \Omega_2 \subsetneqq \Omega_3$, $\Omega_1 \notin \mathbb{A}_3$, $\Omega_2 \notin \mathbb{A}_3$, let $P(Y) = p_3(Y \cap \Omega_3)$ be the probability of hitting a region Y on the second screen, let $P(Y|i) = p_i(Y \cap \Omega_i)/p_i(\Omega_i)$ be the probability of hitting Y when only the slit i is open, $i = 1, 2$. Clearly, in general, $P(Y) = p_3(Y)$ and $P(Y|1) + P(Y|2) = p_1(Y \cap \Omega_1) + p_2(Y \cap \Omega_2)$ can differ (Accardi, n.d.).

3. SEQUENTIAL CONTINUITY

In this section we introduce a new category, *DS*. We claim that *DS* is a suitable category for the study of observables and probabilities within the Kolmogorovian model and the results can be extended in a natural way via Δ-sums to the generalized observables and probabilities.

From the viewpoint of category theory, a probability measure fails to be a morphism, i.e., a map "preserving the structure." As pointed out by F. Chovanec and F. Kôpka, this formal shortcoming disappears within the realm of *D*-posets. A probability measure on a σ-field of sets is a σ-*D*-homomorphism into [0, 1]. We claim that it is natural to consider both observables and probability measures as sequentially continuous *D*-homomorphisms.

Clearly, *D*-posets and *D*-homomorphisms form a concrete category $\mathcal{D}$. Not to destroy completeness and cocompleteness of the categories to be dealt with, we do not exclude from our considerations the *D*-poset for which $0 = 1$ and likewise the field of sets for which the underlying set is empty. To avoid pathologies, we always assume that all fields of sets are reduced. Each field of sets $(\Omega, \mathbb{A})$ carries a natural *D*-poset structure and a sequential convergence: $\langle A_n \rangle$ converges to A in $\mathbb{A}$ iff $A = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n$. It is easy to see that such *D*-posets together with sequentially continuous *D*-homomorphisms form a subcategory of $\mathcal{D}$; denote it by *DS*. Since for fields of sets, *D*-homomorphisms and Boolean homomorphisms coincide, *DS* is isomorphic to the category *FS* of fields of sets and sequentially continuous Boolean homomorphisms studied in Frič (1997, 1999).

Observation 3.1. Let *SCFS* be the subcategory of *FS* consisting of σ-fields of sets and let *SCDS* be the subcategory of *DS* consisting of σ-*D*-posets $(\mathbb{A}, \subseteq, \ominus_{\mathbb{A}}, \emptyset, \Omega)$ such that $(\Omega, \mathbb{A})$ is a σ-field. Since *SCFS* is epireflective in *FS* (Frič, 1997), also *SCDS* is epireflective in *DS*. In particular, this means that each sequentially continuous *D*-homomorphism between fields of sets can be uniquely extended to a sequentially continuous *D*-homomorphism between the generated σ-fields.

Lemma 3.2. Let $(\Omega, \mathbb{A})$ be a field of sets, let $(\mathbb{A}, \subseteq, \ominus_{\mathbb{A}}, \emptyset, \Omega)$ be the corresponding *D*-poset, and let p be a probability measure on $\mathbb{A}$. Then p is a sequentially continuous *D*-homomorphism of $\mathbb{A}$ into [0, 1].

Proof. The assertion follows from the fact that each bounded σ-additive measure on a ring of sets is sequentially continuous (Novák, 1958).

Recall that a field $\mathbb{A}$ of subsets of Ω is *s-perfect* if each maximal filter of sets in $\mathbb{A}$ having the *CIP* (countable intersection property) is generated by a point or, equivalently, each sequentially continuous Boolean homomorphism of $\mathbb{A}$ into the two-element Boolean algebra is generated by a point. Each *perfect* field is *s*-perfect, but the converse does not hold.

Lemma 3.3. For $i = 1, 2$, let $(\Omega_i, \mathbb{A}_i)$ be a σ-field of sets and let $(\mathbb{A}_i, \subseteq, \ominus_{\mathbb{A}_i}, \emptyset, \Omega_i)$ be the corresponding σ-D-poset.

(i) Let f be a measurable map of $(\Omega_2, \mathbb{A}_2)$ into $(\Omega_1, \mathbb{A}_1)$. Then $f^{\leftarrow}$ is a σ-D-homomorphism of $(\mathbb{A}_1, \subseteq, \ominus_{\mathbb{A}_1}, \emptyset, \Omega_1)$ into $(\mathbb{A}_2, \subseteq, \ominus_{\mathbb{A}_2}, \emptyset, \Omega_2)$.

(ii) Let h be a σ-D-homomorphism of $\mathbb{A}_1$ into $\mathbb{A}_2$. Then h is a sequentially continuous Boolean homomorphism of $(\Omega_1, \mathbb{A}_1)$ into $(\Omega_2, \mathbb{A}_2)$. If $\mathbb{A}$ is s-perfect, then there is a unique measurable map f of $(\Omega_2, \mathbb{A}_2)$ into $(\Omega_1, \mathbb{A}_1)$ such that $h = f^{\leftarrow}$.

Proof. (i) Clearly, $f^{\leftarrow}$ is a D-homomorphism. A straightforward calculation shows (cf. Proposition 2.6 in Frič, 1999) that $f^{\leftarrow}$ is sequentially continuous. If $\langle A_n \rangle$ is a nondecreasing sequence in $\mathbb{A}_1$, then $\langle A_n \rangle$ converges to $\cup_{n=1}^{\infty} A_n$ and hence $f^{\leftarrow}(\cup_{n=1}^{\infty} A_n) = \cup_{n=1}^{\infty} f^{\leftarrow}(A_n)$. Hence $f^{\leftarrow}$ is a σ-D-homomorphism.

(ii) Since h is a Boolean homomorphism of $\mathbb{A}_1$ into $\mathbb{A}_2$, it suffices to prove that h is sequentially continuous. First, let $\langle M_n \rangle$ be a monotone sequence in $\mathbb{A}_1$ converging to M. Then either $\langle M_n \rangle$ or $\langle \Omega_1 \ominus_{\mathbb{A}_1} M_n \rangle$ is nondecreasing. Since h is a σ-D-homomorphism, necessarily $\langle h(M_n) \rangle$ converges to $h(M)$. Second, let $\langle A_n \rangle$ be an arbitrary sequence in $\mathbb{A}_1$ converging to A. We are to prove that the sequence $\langle h(A_n) \rangle$ converges in $\mathbb{A}_2$ to $h(A)$, i.e., $h(A) = \cap_{k=1}^{\infty} \cup_{n=k}^{\infty} h(A_n) = \cup_{n=1}^{\infty} \cap_{n=k}^{\infty} h(A_n)$. For $k, l \in N$, put $B_{kl} = \cup_{n=k}^{k-1+l} A_n$ and $C_{kl} = \cup_{n=k}^{k-1+l} A_n$. For fixed k, $\langle B_{kl} \rangle$ and $\langle C_{kl} \rangle$ are monotone sequences converging in $\mathbb{A}_1$ to $B_k = \cup_{n=k}^{\infty} A_n$ and $C_k = \cap_{n=k}^{\infty} A_n$, respectively. But $h(B_{kl}) = \cup_{n=k}^{k-1+l} h(A_n)$ and $h(C_{kl}) = \cap_{n=k}^{k-1+l} h(A_n)$ and hence $h(B_k) = \cup_{n=k}^{\infty} h(A_n)$ and $h(C_k) = \cap_{n=k}^{\infty} h(A_n)$. Finally, from $h(\cap_{k=1}^{n} B_k) = \cap_{k=1}^{n} h(B_n)$ and $h(\cup_{k=1}^{n} C_k) = \cup_{k=1}^{n} h(C_k)$ it follows that $h(A) = \cap_{k=1}^{\infty} \cup_{n=k}^{\infty} h(A_n) = \cup_{k=1}^{\infty} \cap_{n=k}^{\infty} h(A_n)$. The last assertion is now exactly (ii) of Proposition 2.6 in Frič (1999). This completes the proof.

Example 3.4. Consider the minimal field $\mathbb{B}_0$ of subsets of R generated by all intervals of the form $[a, \infty)$, $a \in R$, and the generated σ-field $\mathbb{B}$ of all Borel subsets of R. Denote by δ_0, the Dirac measure concentrated at 0, i.e., $\delta_0(A) = 1$ if $0 \in A$ and $\delta_0(A) = 0$ if $0 \notin A$, $A \in \mathbb{B}$. Then $[1/n, 1) \subset [1/(n+1), 1)$ for each $n \in N$ and $[0, 1) = \vee_{n=1}^{\infty} [1/n, 1)$ in $\mathbb{B}_0$, while $(0, 1) = \vee_{n=1}^{\infty} [1/n, 1)$ in $\mathbb{B}$. Consequently, while δ_0 is a σ-D-homomorphism of $\mathbb{B}$ into $[0, 1]$ (or into the two-element field of sets $\{\emptyset, \Omega\}$), δ_0 restricted to $\mathbb{B}_0$ fails to be a σ-D-homomorphism since $\delta_0([0, 1)) = 1$ and $\delta_0([1/n, 1)) = 0$. Observe that δ_0 is sequentially continuous both on $\mathbb{B}_0$ and on $\mathbb{B}$.

Construction 3.5. Let $\{(\Omega_t, \mathbb{A}_t);\ t \in T\}$ be a family of fields of sets, let $(\mathbb{S}, \subseteq, \ominus_S, \emptyset, \Omega) = \Delta_{t \in T}(\Omega_t, \mathbb{A}_t)$, and let $\{\kappa_t\colon \mathbb{A}_t \to \mathbb{S};\ t \in T\}$ be the corresponding coprojections. Define a sequential convergence on $\mathbb{S}$ as follows: a sequence $\langle A_n \rangle$ converges to A iff there exist $t \in T$, $B \in \mathbb{A}_t$, and a sequence $\langle B_n \rangle$ in $\mathbb{A}_t$ converging to B such that $\kappa_t(B) = A$ and $\kappa_t(B_n) = A_n$ for all but finitely many $n \in N$. Clearly, it is the finest sequential convergence on $\mathbb{S}$, satisfying the Urysohn axiom of convergence, such that all κ_t, $t \in T$, are sequentially continuous. It will be called the *fine* convergence and, in what follows, sequential continuity in a Δ-sum is always with respect to the fine convergence.

Proposition 3.6. Let $\{(X_t, \mathbb{U}_t); t \in T\}$, $\{(Y_t, \mathbb{V}_t); t \in T\}$ be two families of fields of sets and let $\{h_t: \mathbb{U}_t \to \mathbb{V}_t; t \in T\}$ be a family of sequentially continuous D-homomorphisms. Let $(\mathbb{U}, \subseteq, \ominus_{\mathbb{U}}, \emptyset, U) = \Delta_{t \in T}(X_t, \mathbb{U}_t)$, $(\mathbb{V}, \subseteq, \ominus_{\mathbb{V}}, \emptyset, V) = \Delta_{t \in T}(Y_t, \mathbb{V}_t)$ and let $\{\varphi_t: \mathbb{U}_t \to \mathbb{U}; t \in T\}$, $\{\psi_t: \mathbb{V}_t, \to \mathbb{V}; t \in T\}$ be the corresponding coprojections. Assume that each $\mathbb{U}_t$, $t \in T$, is s-perfect.

(i) There exists a unique D-homomorphism $h: \mathbb{U} \to \mathbb{V}$ such that $h \circ \varphi_t = \psi_t \circ h_t$.
(ii) There exists a unique map $f: Y \to X$ such that $f^{\nabla} = h$.
(iii) h is sequentially continuous.

Proof. (i) follows from the fact that $\mathbb{U}$ together with $\{\varphi_t: \mathbb{U}_t \to \mathbb{U}; t \in T\}$ is the coproduct of $\{(X_t, \mathbb{U}_t); t \in T\}$.

(ii) follows from the fact that for each $t \in T$, there exists a unique measurable map f_t of $(Y_t, \mathbb{V}_t)$ into $(X_t, \mathbb{U}_t)$ such that $f_t^{\leftarrow} = h_t$ (cf. Proposition 2.6 in Frič, 1999), the construction of a Δ-sum, and the construction of the generalized preimage f^{∇} of f.

(iii) is straightforward. This completes the proof.

Denote by *MM* the category objects of which are fields of sets and morphisms of which are measurable maps. The most important result about s-perfectness is that the subcategory *SPMM* consisting of s-perfect objects in *MM* and the subcategory *SPFS* consisting of s-perfect objects in *FS* are dually isomorphic (Frič, 1999). Since each measurable map f induces a sequentially continuous Boolean homomorphism $f^{\leftarrow}$, the duality means that, for s-perfect fields of sets, each sequentially continuous Boolean homomorphism is induced by a measurable map. In fact, it suffices that the domain of the homomorphism is s-perfect. Further, s-perfectness is preserved under the products of fields of sets and the generation of σ-field. Since $\mathbb{B}_0$ is s-perfect, for each power R^T the field $\mathbb{B}^T$ of all Borel sets in R^T is s-perfect, too. Thus each sequentially continuous Boolean homomorphism of $(R^T, \mathbb{B}^T)$ into a σ-field of sets $(\Omega, \mathbb{A})$ is induced by a measurable map of $(\Omega, \mathbb{A})$ into $(R^T, \mathbb{B}^T)$.

Proposition 3.7. The categories *DS* and *SPMM* are dual.

Proof. Recall that *DS* and *FS* are isomorphic categories. Our proof is based on the following facts from Frič (1999).

(i) *SPFS* and *SPMM* are dually isomorphic (cf. Proposition 3.2 in Frič, 1999), hence it suffices to show that *FS* and *SPFS* are equivalent categories.

(ii) Let $(\Omega, \mathbb{A})$ be a field of sets. Then there exists a unique *s*-perfect field of sets $(\Omega^*, \mathbb{A}^*)$ such that $\Omega \subseteq \Omega^*$, $\mathbb{A} = \{A \cap \Omega | A \in \mathbb{A}^*\}$, and a natural isomorphism h sending $A \in \mathbb{A}^*$ to $A \cap \Omega$ such that both h and h^{-1} are sequentially continuous.

(iii) Putting $F((\Omega, \mathbb{A})) = (\Omega^*, \mathbb{A}^*)$, we get a (bireflective) functor F: $FS \to SPFS$.

(iv) F is a left adjoint to the inclusion functor G: $SPFS \to FS$ and the adjunction is an equivalence (both the unit and the counit of the adjunction are isomorphisms). This completes the proof.

SUMMARY

Since each probability measure on a field of sets $\mathbb{A}$ is sequentially continuous and can be extended to a probability measure on the generated σ-field $\sigma(\mathbb{A})$, it follows from the assertions proved in this section that it is natural to consider each probability measure as a sequentially continuous *D*-homomorphism of a field of sets into [0, 1] and to consider each observable as a sequentially continuous *D*-homomorphism between fields of sets. Let $\{(\Omega_t, \mathbb{A}_t, p_t); t \in T\}$ be a family of classical probability spaces. For each $t \in T$, there is a one-to-one correspondence between random functions $F_t = \{f_{(t,u)}: (\Omega_t, \mathbb{A}_t) \to (R, \mathbb{B}) | u \in U_t\}$ and (multidimensional) observables $h_t: \mathbb{B}^{U_t} \to \mathbb{A}_t$ as sequentially continuous *D*-homomorphisms or, equivalently, as sequentially continuous Boolean homomorphisms. Passing to the corresponding Δ-sums, we get a one-to-one correspondence between generalized random functions as point maps F (represented by families $\{F_t; t \in T\}$) of the underlying set of $\Delta_{t \in T}(\Omega_t, \mathbb{A}_t)$ and generalized observables F^{∇} (represented by families $\{F_t^{\leftarrow}; t \in T\}$) as sequentially continuous *D*-homomorphisms of $\Delta_{t \in T}(R, \mathbb{B})^{U_t}$ into $\Delta_{t \in T}(\Omega_t, \mathbb{A}_t)$. There is a unique sequentially continuous *D*-homomorphism P of $\Delta_{t \in T}(\Omega_t, \mathbb{A}_t)$ into [0, 1] such that $\{p_t; t \in T\}$ is the spectrum of P. Compatibility of events and random functions means that they are "located" in the same $(\Omega_t, \mathbb{A}_t)$.

In a forthcoming paper we shall develop a categorical approach to classical random variables and functions. The results will be applied to generalized random variables via Δ-sums.

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